

Effective Weak Chiral Lagrangian to $\mathcal{O}(p^4)$ in the Chiral Quark Model ^{*}

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Abstract

We investigate the $\Delta S = 1, 2$ effective weak chiral Lagrangian within the framework of the chiral quark model. Starting from the effective four-quark operators, we derive the effective weak chiral action by integrating out the constituent quark fields. Employing the derivative expansion, we obtain the effective weak chiral Lagrangian to order $\mathcal{O}(p^4)$. We examine the contributions of the order $\mathcal{O}(N_c)$ to the ratio g_8/g_{27} , considering e.g. the quark axial-vector constant g_A different from unity. The low energy constants of the counterterms are also presented and discussed.

Keywords: Chiral quark model, Effective chiral Lagrangians, Nonleptonic decays, Derivative expansion.

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I. INTRODUCTION

Processes involving the creation or annihilation of strangeness are described in the Standard Model by W -exchange. While the theoretical formulation is simple at scales around the W -mass the description of nonleptonic decays of light hadrons at low energies is complicated and difficult because of the presence of the strong interaction. The problem is characterized by the $\Delta T = 1/2$ selection rule, best known as the fact that the isospin $T = 0$ amplitude of the $K \rightarrow \pi\pi$ decay is about 22 times larger than the $T = 2$ amplitude. In spite of many efforts this enhancement of the $\Delta T = 1/2$ channel over the $\Delta T = 3/2$ channel has not been explained in a satisfactory way. A part of the answer comes from perturbative gluons which are created if one evolves the simple W -exchange-vertex from a scale of 80 GeV down to 1 GeV [1–9]. Another part of the answer is supposed to arise from the structure of the light hadrons, whose description at scales around 1 GeV requires a nonperturbative QCD-method.

In the low energy regime one way to deal with nonperturbative effects is to utilize the large N_c expansion with $\alpha_s N_c$ fixed (N_c being the number of colors and α_s the running coupling constant of QCD). In the limit of large N_c QCD can be treated as a weakly coupled meson field theory and indeed many experimental consequences have been explained in this way [10,11]. The large N_c limit of QCD was also employed [12–14] in order to understand the $\Delta T = 1/2$ problem in the $K \rightarrow \pi\pi$ decay. However, in contrast to the sector of the pure strong interaction, the large N_c limit in its strict form (only leading order in N_c) does not seem to be sufficient to describe the weak non-leptonic decays [15] because it enhances the $\Delta T = 3/2$ channel while suppressing the $\Delta T = 1/2$ one making the problem even more difficult. Hence for these processes one is bound to go beyond leading order in the large N_c expansion.

At low energies chiral perturbation theory (χ PT) [16] is known as a proper effective field theory of QCD in the mesonic sector. Based on its success in describing strong interactions χ PT was also applied to nonleptonic processes of light mesons [17–19]. However in this case there are not enough experimental data available to determine the many low energy constants (LECs) of the the effective weak chiral Lagrangian to order $\mathcal{O}(p^4)$. Hence, in order to proceed without experiments, one is advised to determine the LECs of the weak chiral Lagrangian by using effective QCD-inspired models.

In the present paper we are going to investigate how far the chiral quark model (χ QM) furnishes a reasonable framework to determine the LECs of the weak chiral Lagrangian. We are motivated to this study by success of the χ QM to determine the LECs of the strong chiral Lagrangian. The χ QM is characterized by the Euclidean partition function [20]

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{D}\pi \exp \left[\int d^4x \psi_f^\dagger \left(i\not{\partial} + iM e^{i\gamma_5 \lambda^a \pi^a} \right)_{fg} \psi_g^\alpha \right], \quad (1)$$

where α is the color index, $\alpha = 1, \dots, N_c$ and f and g are flavor indices. The M serves as the coupling parameter between the constituent quark fields ψ and the Goldstone boson field π^a and it can be identified with the constituent quark mass. In this work we want to construct systematically the effective weak chiral Lagrangian without external fields to order $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ for $\Delta S = 1$ and $\Delta S = 2$. We will show that the χ QM provides the most general structure of the Lagrangian, known from the work of Refs. [17–19], and unique descriptions of the LECs in leading and subleading order of the large N_c expansion. The calculations start from the effective weak Hamiltonians for $\Delta S = 1, 2$ of Refs. [9,21–23].

The effective weak Lagrangian for $\Delta S = 1$ to order $\mathcal{O}(p^2)$ in leading and subleading order in N_c has been given already by Antonelli *et al.* [24]. Bertolini *et al.* [25] extended the former calculation to $\mathcal{O}(p^4)$ in the study of ϵ'/ϵ and \hat{B}_K with the $\Delta S = 1$ Lagrangian, which implies that parts of the $\mathcal{O}(p^4)$ effective chiral weak Lagrangian that are necessary for the description of the $K \rightarrow \pi\pi$ decays are obtained. They used for this the small field expansion in leading and subleading order in the N_c expansion. In the present paper we prefer the derivative expansion, since we want to evaluate the full effective chiral weak Lagrangian to order $\mathcal{O}(p^4)$ including the corresponding LECs in a way that it can be used directly in χ PT. This means we have to use an expansion of the χ QM which is consistent with the chiral expansion in χ PT. In the case of the strong interaction it is known that the small field expansion fulfills this criterion only in the leading order in the large N_c expansion. However, for the weak chiral Lagrangian the subleading order in N_c is necessary and hence the derivative expansion seems to us more appropriate than the small field expansion¹.

The outline of the present paper is as follows: In section 2 we sketch the characteristics of the chiral quark model and briefly show how to use the derivative expansion. In section 3 we review the effective weak chiral action at a scale of 1 GeV and discuss some of its properties relevant for the following. Section 4 is devoted to the derivation of the effective weak chiral Lagrangian to order $\mathcal{O}(p^2)$. We examine the dependence of the LECs on the constituent quark mass, the quark condensate and the quark axial-vector constant. The full effective weak chiral Lagrangian for $\Delta S = 1$ and $\Delta S = 2$ to order $\mathcal{O}(p^4)$ in leading and next-to-leading order in N_c , as it results from the derivative expansion, is presented in Section 5. The conclusions are given in Section 6.

II. CHIRAL QUARK MODEL

The characteristic of the chiral quark model is represented by the effective chiral action in Euclidean space given by the functional integral over quark fields [20]:

$$\mathcal{N} = \exp(-S_{\text{eff}}) = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp \left[\int d^4x \psi_f^\dagger (i\not{\partial} + iMU^{\gamma_5})_{fg} \psi_g \right], \quad (2)$$

where α is the color index, $\alpha = 1, \dots, N_c$ and f and g are flavor indices. M is the constituent quark mass, which is in fact momentum-dependent. However, we regard it as a free parameter for convenience and introduce a cut-off parameter to tame the divergence appearing in the quark loop. It is fixed by producing the pion decay constant. U^{γ_5} denotes the Goldstone field

$$U^{\gamma_5} = \exp(i\pi^a \lambda^a \gamma_5) = U \frac{1 + \gamma_5}{2} + U^\dagger \frac{1 - \gamma_5}{2} \quad (3)$$

¹The LECs L_1 and $L_2/2$ of the Gasser-Leutwyler Lagrangian [16] in the strong interaction are the same in the large N_c limit, which makes the small field expansion in the chiral quark model yield the same Lagrangian as in the derivative expansion. However, when one considers higher order corrections L_1 and $L_2/2$ have to deviate from each other, as the experimental extraction of those values implies, a feature, which is only brought out by the derivative expansion.

with

$$U = \exp(i\pi^a \lambda^a). \quad (4)$$

The π stands for the meson octet fields

$$\pi = \pi^a \lambda^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta \end{pmatrix}. \quad (5)$$

Integrating over the quark fields in Eq.(2), we obtain the following expression for the effective chiral action

$$S_{\text{eff}}[U] = -N_c \text{Tr} \ln D \quad (6)$$

where Tr designates the functional trace as well as flavor and spin ones. The D denotes the Dirac operator

$$D = i\not{\partial} + iMU^{\gamma_5}. \quad (7)$$

Since Eq.(6) is non-Hermitian, one can separate the effective action into the real part and the imaginary one. The real part can be written as

$$\text{Re}S_{\text{eff}}[U] = -\frac{1}{2} \text{Tr} \ln \left(\frac{D^\dagger D}{D_0^\dagger D_0} \right), \quad (8)$$

where

$$\begin{aligned} D^\dagger D &= -\partial^2 + M^2 - M(\not{\partial}U^{\gamma_5}), \\ D_0^\dagger D_0 &= -\partial^2 + M^2. \end{aligned} \quad (9)$$

It is already well known how to treat the effective action in order to obtain the effective chiral Lagrangian [26–29]. The real part of the effective action may be expanded with respect to the derivatives of the meson field [20]:

$$\begin{aligned} \text{Re}S_{\text{eff}} &= -\frac{N_c}{2} \text{Tr} \ln \left(1 - \frac{M(\not{\partial}U^{\gamma_5})}{D_0^\dagger D_0} \right) \\ &= -\frac{N_c}{2} \text{tr} \int d^4x \left\langle x \left| \ln \left(1 - \frac{M(\not{\partial}U^{\gamma_5})}{D_0^\dagger D_0} \right) \right| x \right\rangle \\ &= -\frac{N_c}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{tr} \ln \left(1 - \frac{M(\not{\partial}U^{\gamma_5})}{k^2 + M^2 - (2ik \cdot \partial + \partial^2)} \right) \cdot 1 \end{aligned} \quad (10)$$

The nonvanishing leading term in the expansion is just the kinetic Lagrangian of the strong interaction:

$$\text{Re}S_{\text{eff}}^{(2)}[U] = - \int d^4x \mathcal{L}_{\Delta S=0}^{(2)}, \quad (11)$$

where

$$\mathcal{L}_{\Delta S=0}^{(2)} = -\frac{f_\pi^2}{4} \langle L_\mu L_\mu \rangle \quad (12)$$

The L_μ (R_μ) are the Noether currents of $SU(3)_L \times SU(3)_R$ chiral symmetry:

$$L_\mu = iU^\dagger \partial_\mu U, \quad R_\mu = iU \partial_\mu U^\dagger. \quad (13)$$

The symbol $\langle \rangle$ stands for the flavor trace. The f_π denote the pion decay constant which is related to the following quark loop integral:

$$f_\pi^2 = 4N_c \int \frac{d^4 k}{(2\pi)^4} \frac{M^2}{(k^2 + M^2)^2}. \quad (14)$$

Since the quark loop integral is divergent, which is due to the fact that we regard the M as a constant, we need to introduce the cut-off parameter Λ via regularization. It is fixed by producing the experimental value of $f_\pi = 93$ MeV.

Similarly, we can move up to higher orders in the derivative expansion. The real part of the effective chiral action in the next-to-leading order $\mathcal{O}(p^4)$ is given by

$$\text{Re} S_{\text{eff}}^{(4)} = - \int d^4 x \mathcal{L}_{\Delta S=0}^{(4)}, \quad (15)$$

so that the strong effective chiral Lagrangian to order $\mathcal{O}(p^4)$ can be written as [20]

$$\mathcal{L}_{\Delta S=0}^{(4)} = \frac{N_c}{192\pi^2} \int d^4 x \left[2\langle (\partial_\mu L_\mu)^2 \rangle + \langle L_\mu L_\nu L_\mu L_\nu \rangle \right]. \quad (16)$$

Those effective Lagrangians in higher orders were extensively studied [26–32]. Ref. [31] investigated also the low energy constants of the effective $\mathcal{O}(p^4)$ Lagrangian in relation to chiral perturbation theory.

The imaginary part of the effective chiral action is pertinent to the Wess-Zumino-Witten(WZW) action [33,34] with the correct coefficient, which arises from the derivative expansion of the imaginary part to order $\mathcal{O}(p^5)$ (see Ref. [20] for details.).

III. EFFECTIVE WEAK CHIRAL ACTION

The effective chiral action in Eq.(1) with the weak $\Delta S = 1$ or $\Delta S = 1$ effective Hamiltonian can be written as follows:

$$\exp \left(-S_{\text{eff}}^{\Delta S=1,2} \right) = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp \left[\int d^4 x \left(\psi^\dagger D\psi - \mathcal{H}_{\text{eff}}^{\Delta S=1,2} \right) \right], \quad (17)$$

Here the effective weak quark Hamiltonian $\mathcal{H}_{\text{eff}}^{\Delta S=1}$ consists of ten four-quark operators among which only seven operators are independent

$$\mathcal{H}_{\text{eff}}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_i c_i(\mu) \mathcal{Q}_i(\mu) + \text{h.c.} \quad (18)$$

The G_F is the well-known Fermi constant and V_{ij} denote the Cabibbo-Kobayashi-Maskawa(CKM) matrix elements. The τ is their ratio given by $\tau = -V_{td} V_{ts}^* / V_{ud} V_{us}^*$. The

$c_i(\mu)$ consist of the Wilson coefficients: $c_i(\mu) = z_i(\mu) + \tau y_i(\mu)$. The functions $z_i(\mu)$ and $y_i(\mu)$ are the scale-dependent Wilson coefficients given at the scale of the μ . The $z_i(\mu)$ represent the CP -conserving part, while $y_i(\mu)$ stand for the CP -violating one. The four-quark operators \mathcal{Q}_i contain the dynamic information of the weak transitions, being constructed by integrating out the vector bosons W^\pm and Z and heavy quarks t , b and c . The four-quark operators [9] are given by

$$\mathcal{Q}_1 = 4 \left(s_\alpha^\dagger \gamma_\mu P_L u_\beta \right) \left(u_\beta^\dagger \gamma_\mu P_L d_\alpha \right), \quad (19)$$

$$\mathcal{Q}_2 = 4 \left(s_\alpha^\dagger \gamma_\mu P_L u_\alpha \right) \left(u_\beta^\dagger \gamma_\mu P_L d_\beta \right), \quad (20)$$

$$\mathcal{Q}_3 = 4 \left(s_\alpha^\dagger \gamma_\mu P_L d_\alpha \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \gamma_\mu P_L q_\beta \right), \quad (21)$$

$$\mathcal{Q}_4 = 4 \left(s_\alpha^\dagger \gamma_\mu P_L d_\beta \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \gamma_\mu P_L q_\alpha \right), \quad (22)$$

$$\mathcal{Q}_5 = 4 \left(s_\alpha^\dagger \gamma_\mu P_L d_\alpha \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \gamma_\mu P_R q_\beta \right), \quad (23)$$

$$\mathcal{Q}_6 = 4 \left(s_\alpha^\dagger \gamma_\mu P_L d_\beta \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \gamma_\mu P_R q_\alpha \right), \quad (24)$$

$$\mathcal{Q}_7 = 6 \left(s_\alpha^\dagger \gamma_\mu P_L d_\alpha \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \hat{Q} \gamma_\mu P_R q_\beta \right), \quad (25)$$

$$\mathcal{Q}_8 = 6 \left(s_\alpha^\dagger \gamma_\mu P_L d_\beta \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \hat{Q} \gamma_\mu P_R q_\alpha \right), \quad (26)$$

$$\mathcal{Q}_9 = 6 \left(s_\alpha^\dagger \gamma_\mu P_L d_\alpha \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \hat{Q} \gamma_\mu P_L q_\beta \right), \quad (27)$$

$$\mathcal{Q}_{10} = 6 \left(s_\alpha^\dagger \gamma_\mu P_L d_\beta \right) \sum_{q=u,d,s} \left(q_\beta^\dagger \hat{Q} \gamma_\mu P_L q_\alpha \right), \quad (28)$$

where $P_{L,R} = \frac{1}{2} (1 \pm \gamma_5)$ are the chiral projection operators and $\hat{Q} = \frac{1}{3} \text{diag}(2, -1, -1)$ denote the quark charge matrix. The \mathcal{Q}_1 and \mathcal{Q}_2 come from the current-current diagrams, while \mathcal{Q}_3 to \mathcal{Q}_6 [5–7] and \mathcal{Q}_7 to \mathcal{Q}_{10} [8] are induced by QCD penguin and electroweak penguin diagrams, respectively. Note that only seven operators in Eqs.(19-28) are independent. For example, we can express \mathcal{Q}_4 , \mathcal{Q}_9 , and \mathcal{Q}_{10} as follows:

$$\mathcal{Q}_4 = -\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3, \quad \mathcal{Q}_9 = \frac{1}{2} (3\mathcal{Q}_1 - \mathcal{Q}_3), \quad \mathcal{Q}_{10} = \mathcal{Q}_2 + \frac{1}{2} (\mathcal{Q}_1 - \mathcal{Q}_3). \quad (29)$$

Under the chiral transformation $SU(3)_L \times SU(3)_R$ the four-quark operators $\mathcal{Q}_{3,4,5,6}$ transform like $(\underline{8}_L, \underline{1}_R)$. The $\mathcal{Q}_{1,2,9,10}$ transform like the combination of $(\underline{8}_L, \underline{1}_R)$ and $(\underline{27}_L, \underline{1}_R)$, while the $\mathcal{Q}_{7,8}$ transform like $(\underline{8}_L, \underline{8}_R)$. The $\Delta S = 2$ effective weak Hamiltonian is expressed as [35,21–23]

$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = -\frac{G_F^2 M_W^2}{16\pi^2} \mathcal{F} \left(\lambda_c, \lambda_t, m_c^2, m_t^2, M_W^2 \right) b(\mu) \mathcal{Q}_{\Delta S=2}(\mu) + \text{h.c.} \quad (30)$$

with

$$\mathcal{F} = \lambda_c^2 \eta_1 S \left(\frac{m_c^2}{M_W^2} \right) + \lambda_t^2 \eta_2 S \left(\frac{m_t^2}{M_W^2} \right) + 2\lambda_c \lambda_t \eta_3 S \left(\frac{m_c^2}{M_W^2}, \frac{m_t^2}{M_W^2} \right) \quad (31)$$

and the parameters $\lambda_q = V_{qd}V_{qs}^*$ denote the pertinent relations of the CKM matrix elements with $q = u, c, t$. The functions S_i are the Inami-Lim functions [35–37], being obtained by integrating over electroweak loops and describing the $|\Delta S| = 2$ transition amplitude in the absence of strong interactions. The $b(\mu)$ is again the corresponding Wilson coefficient. The coefficients η_i represent the short-distance QCD corrections split off from the $b(\mu)$ [23]. The four-quark operator $\mathcal{Q}_{\Delta S=2}$ is written as

$$\mathcal{Q}_{\Delta S=2} = 4 \left(s_\alpha^\dagger \gamma_\mu P_L d_\alpha \right) \left(s_\beta^\dagger \gamma_\mu P_L d_\beta \right). \quad (32)$$

Since the Fermi constant G_F is very small, one can expand Eq.(17) in powers of the G_F and keep the lowest order only. Then we can obtain the effective weak chiral Lagrangian

$$\mathcal{L}_{\text{eff}}^{\Delta S=1,2} = -\frac{1}{\mathcal{N}} \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{H}_{\text{eff}}^{\Delta S=1,2} \exp \left[\int d^4x \psi^\dagger D\psi \right]. \quad (33)$$

If you write a generic operator for the four-quark operator \mathcal{Q}_i for a given i in Euclidean space such as

$$\mathcal{Q}_i(x) = \psi^\dagger(x) \gamma_\mu P_{R,L} \Lambda_1 \psi(x) \psi^\dagger(x) \gamma_\mu P_{R,L} \Lambda_2 \psi(x), \quad (34)$$

where $\Lambda_{1,2}$ denote the flavor spin operators, then we can calculate the vacuum expectation value (VEV) of $\mathcal{Q}_i(x)$ as follows:

$$\begin{aligned} \langle \mathcal{Q}_i \rangle &= \frac{1}{\mathcal{N}} \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{Q}_i(x) \exp \left[\int d^4z \psi^\dagger D\psi \right] \\ &= \int d^4y \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{\delta}{\delta J_\mu^{(1)}(x)} \frac{\delta}{\delta J_\mu^{(2)}(y)} \\ &\quad \times \exp \left[\int d^4z \langle z | \text{tr} \ln \tilde{D}(J_1(z), J_2(z)) | z \rangle \right]_{J_1=J_2=0} \\ &= L_i^{(1)} + L_i^{(2)}. \end{aligned} \quad (35)$$

Here, \tilde{D} is

$$\tilde{D}(J_1(z), J_2(z)) = D + J_\alpha^{(1)}(z) \gamma_\alpha P_{R,L} \Lambda_1 + J_\beta^{(2)}(z) \gamma_\beta P_{R,L} \Lambda_2. \quad (36)$$

The $L_i^{(1)}$ and $L_i^{(2)}$ are given by

$$\begin{aligned} L_i^{(1)} &= -N_c^2 \text{tr} \left[\left\langle x \left| \frac{1}{D} \gamma_\mu P_{R,L} \Lambda_1 \right| x \right\rangle \left\langle x \left| \frac{1}{D} \gamma_\mu P_{R,L} \Lambda_2 \right| x \right\rangle \right]_i + \mathcal{O}(N_c) \\ &= -N_c^2 \text{tr} \left[(A_1)_\mu (A_2)_\mu \right]_i + \mathcal{O}(N_c) \quad i = 1, 4, 6, 8, 10, \end{aligned} \quad (37)$$

$$\begin{aligned} L_i^{(2)} &= N_c^2 \text{tr} \left[\left\langle x \left| \frac{1}{D} \gamma_\mu P_{R,L} \Lambda_1 \right| x \right\rangle \right] \text{tr} \left[\left\langle x \left| \frac{1}{D} \gamma_\mu P_{R,L} \Lambda_2 \right| x \right\rangle \right]_i + \mathcal{O}(N_c) \\ &= N_c^2 \text{tr} \left[(A_1)_\mu \right] \text{tr} \left[(A_2)_\mu \right]_i + \mathcal{O}(N_c) \quad i = 2, 3, 5, 7, 9, \end{aligned} \quad (38)$$

where $\Lambda_{1,2}$ are the corresponding flavor matrices. The operators $(A_{1,2})_\mu$ can be written as

$$(A_{1,2})_\mu = \int \frac{d^4k}{(2\pi)^4} \left[\frac{i \not{\partial} + \not{k} - i M U^{-\gamma_5}}{k^2 + M^2 - \partial^2 + 2ik \cdot \partial - M(\not{\partial} U \gamma_5)} \right] \gamma_\mu P_{L,R} \Lambda_{1,2}. \quad (39)$$

Assuming that the pion field changes adiabatically, we are able to expand the denominator in Eq.(39) in powers of $\partial^2 - 2ik \cdot \partial + M(\not{\partial}U^{\gamma_5})$ and obtain the following expression:

$$(A_{1,2})_\mu = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + M^2} \sum_{n=0}^{\infty} \left[\frac{\partial^2 - 2ik \cdot \partial + M(\not{\partial}U^{\gamma_5})}{k^2 + M^2} \right]^n \times (i \not{\partial} + \not{k} - iMU^{-\gamma_5}) \gamma_\mu P_{L,R} \Lambda_{1,2}. \quad (40)$$

With the expansion given in Eq.(40) we can systematically evaluate effective weak chiral Lagrangian to order $\mathcal{O}(p^4)$:

$$\mathcal{L}_{\text{eff}}^{\Delta S=1,2} = \mathcal{L}_{\text{eff}}^{\Delta S=1,2}(\mathcal{O}(p^2)) + \mathcal{L}_{\text{eff}}^{\Delta S=1,2}(\mathcal{O}(p^4)). \quad (41)$$

We first evaluate the effective weak chiral Lagrangian in the lowest order.

IV. LOWEST ORDER P^2 AND LOW ENERGY CONSTANTS

A. Leading order in the $1/N_c$ expansion

The derivation of the $\mathcal{L}_{\text{eff}}^{\Delta S=1,2}(\mathcal{O}(p^2))$ is straightforward. At lowest leading order in the derivative expansion, *i.e.* $\mathcal{O}(p^2)$ order, we obtain the following results with $\mathcal{O}(N_c^2)$ considered:

$$\langle \mathcal{Q}_1 + \mathcal{Q}_1^\dagger \rangle_{\mathcal{O}(p^2)} = f_\pi^4 \left(-\frac{2}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{3} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right), \quad (42)$$

$$\langle \mathcal{Q}_2 + \mathcal{Q}_2^\dagger \rangle_{\mathcal{O}(p^2)} = f_\pi^4 \left(\frac{3}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{3} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right), \quad (43)$$

$$\langle \mathcal{Q}_3 + \mathcal{Q}_3^\dagger \rangle_{\mathcal{O}(p^2)} = 0, \quad (44)$$

$$\langle \mathcal{Q}_4 + \mathcal{Q}_4^\dagger \rangle_{\mathcal{O}(p^2)} = f_\pi^4 \langle \lambda_6 L_\mu L_\mu \rangle, \quad (45)$$

$$\langle \mathcal{Q}_5 + \mathcal{Q}_5^\dagger \rangle_{\mathcal{O}(p^2)} = 0, \quad (46)$$

$$\langle \mathcal{Q}_6 + \mathcal{Q}_6^\dagger \rangle_{\mathcal{O}(p^2)} = \left(\frac{\langle \bar{q}q \rangle f_\pi^2}{M} - \frac{\langle \bar{q}q \rangle N_c M}{8\pi^2} \right) f_\pi^4 \langle \lambda_6 L_\mu L_\mu \rangle, \quad (47)$$

$$\langle \mathcal{Q}_7 + \mathcal{Q}_7^\dagger \rangle_{\mathcal{O}(p^2)} = \frac{3}{2} f_\pi^4 \langle L_\mu \lambda_6 \rangle \langle R_\mu \hat{Q} \rangle, \quad (48)$$

$$\begin{aligned} \langle \mathcal{Q}_8 + \mathcal{Q}_8^\dagger \rangle_{\mathcal{O}(p^2)} = & - \left(\frac{N_c \langle \bar{q}q \rangle M}{16\pi^2} + \frac{f_\pi^2 \langle \bar{q}q \rangle}{2M} \right) \left[\langle U \lambda_6 (\partial^2 U^\dagger) \hat{Q} \rangle + \langle (\partial^2 U) \lambda_6 U^\dagger \hat{Q} \rangle \right] \\ & - \frac{N_c \langle \bar{q}q \rangle M}{8\pi^2} \left[\langle U \lambda_6 (\partial_\mu U^\dagger) (\partial_\mu U) U^\dagger \hat{Q} \rangle + \langle (\partial_\mu U) (\partial_\mu U^\dagger) U \lambda_6 U^\dagger \hat{Q} \rangle \right], \end{aligned} \quad (49)$$

$$\langle \mathcal{Q}_9 + \mathcal{Q}_9^\dagger \rangle_{\mathcal{O}(p^2)} = f_\pi^4 \left(-\frac{3}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{2} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right), \quad (50)$$

$$\langle \mathcal{Q}_{10} + \mathcal{Q}_{10}^\dagger \rangle_{\mathcal{O}(p^2)} = f_\pi^4 \left(\frac{2}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{2} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right), \quad (51)$$

$$\langle \mathcal{Q}_{\Delta S=2} + \mathcal{Q}_{\Delta S=2}^\dagger \rangle_{\mathcal{O}(p^2)} = f_\pi^4 \langle \lambda_6 L_\mu \lambda_6 L_\mu \rangle. \quad (52)$$

The eikosiheptaplet projection operators $t_{ij;kl}$ are defined by

$$t_{ij;kl} = \frac{1}{5} t_{ij;kl}^{T=1/2} + t_{ij;kl}^{T=3/2}, \quad (53)$$

where

$$\begin{aligned} t_{13;21}^{T=1/2} &= t_{31;12}^{T=1/2} = t_{21;13}^{T=1/2} = t_{12;31}^{T=1/2} = \frac{1}{2}, \\ t_{23;11}^{T=1/2} &= t_{32;11}^{T=1/2} = t_{11;23}^{T=1/2} = t_{11;32}^{T=1/2} = \frac{1}{2}, \\ t_{23;22}^{T=1/2} &= t_{32;22}^{T=1/2} = t_{22;23}^{T=1/2} = t_{22;32}^{T=1/2} = 1, \\ t_{23;33}^{T=1/2} &= t_{32;33}^{T=1/2} = t_{33;23}^{T=1/2} = t_{33;32}^{T=1/2} = -\frac{3}{2}, \\ t_{13;21}^{T=3/2} &= t_{31;12}^{T=3/2} = t_{21;13}^{T=3/2} = t_{12;31}^{T=3/2} = \frac{1}{2}, \\ t_{13;21}^{T=3/2} &= t_{31;12}^{T=3/2} = t_{21;13}^{T=3/2} = t_{12;31}^{T=3/2} = \frac{1}{2}, \\ t_{13;21}^{T=3/2} &= t_{31;12}^{T=3/2} = t_{21;13}^{T=3/2} = t_{12;31}^{T=3/2} = -\frac{1}{2}, \\ t_{ij;kl}^{T=1/2} &= t_{ij;kl}^{T=3/2} = 0, \quad \text{for the other } i, j, k, l, \end{aligned} \quad (54)$$

and

$$(\lambda_{ij})_{ab} = \delta_{ia} \delta_{ib}. \quad (55)$$

The coefficients appearing in front of the integrals consist of the pion decay constant f_π (see Eq.(14)), quark condensate² $\langle \bar{q}q \rangle$, and constituent quark mass M . The quark condensate is related to the following quadratically-divergent integral:

$$\langle \bar{q}q \rangle = 8N_c \int \frac{d^4k}{(2\pi)^4} \frac{M}{k^2 + M^2}. \quad (56)$$

Hence, we find that those coefficients have $\mathcal{O}(N_c^2)$ order in the N_c counting. Note that the VEV of the operators $\langle \mathcal{Q}_3 \rangle$ and $\langle \mathcal{Q}_5 \rangle$ vanish at leading order in N_c , which implies that in the leading order of the large N_c expansion $\langle \mathcal{Q}_3 \rangle$ and $\langle \mathcal{Q}_5 \rangle$ do not contribute to the effective weak chiral Lagrangian in the order $\mathcal{O}(p^2)$.

It is interesting to compare our results with those of Ref. [24]. We find some differences in $\langle \mathcal{Q}_6 \rangle$ and $\langle \mathcal{Q}_8 \rangle$ which, however, disappear when we apply for the quark condensate the same regularization scheme as used in Ref. [24]. Since Ref. [24] employs the expansion of the weak meson field in which the U field is expanded in powers of the π field, one is not able to obtain the full effective chiral Lagrangian to order $\mathcal{O}(p^4)$ consistently with the chiral expansion, if one goes beyond the leading order in the large N_c expansion. With the derivative expansion, we can derive the effective weak chiral Lagrangian in the next-to-leading order $\mathcal{O}(p^4)$. In fact, the VEV of the operator \mathcal{Q}_8 has the zeroth order contribution:

² In our calculation the quark condensate is defined as $\langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d \rangle$, since it plays the role of a numerical parameter in our calculation we do not distinguish between the quark condensate in Euclidean and Minkowski space.

$$\langle \mathcal{Q}_8 + \mathcal{Q}_8^\dagger \rangle_{\mathcal{O}(p^0)} = \frac{3}{4} \langle \bar{q}q \rangle^2 \langle \lambda_6 U^\dagger \hat{Q} U \rangle. \quad (57)$$

It transforms under $SU(3)_L \times SU(3)_R$ as $(\underline{8}_L, \underline{8}_R)$.

The effective weak chiral Lagrangian describing the $\Delta S = 1$ nonleptonic decays of kaons was first introduced by Cronin [38] (presented in Minkowski space):

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{\Delta S=1, \mathcal{O}(p^2)} &= -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* f_\pi^4 \left[g_{\underline{8}} \langle \lambda_{23} L_\mu L^\mu \rangle \right. \\ &\quad \left. + g_{\underline{27}} \left(\frac{2}{3} \langle \lambda_{12} L_\mu \rangle \langle \lambda_{31} L^\mu \rangle + \langle \lambda_{32} L_\mu \rangle \langle \lambda_{11} L^\mu \rangle \right) \right] + \text{h.c.} \\ &= \mathcal{L}_{\underline{8}}^{(1/2)} + \frac{1}{9} \mathcal{L}_{\underline{27}}^{(1/2)} + \frac{5}{9} \mathcal{L}_{\underline{27}}^{(3/2)}, \end{aligned} \quad (58)$$

where

$$\begin{aligned} \mathcal{L}_{\underline{8}}^{(1/2)} &= -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* f_\pi^4 g_{\underline{8}} \langle \lambda_{23} L_\mu L^\mu \rangle + \text{h.c.}, \\ \mathcal{L}_{\underline{27}}^{(1/2)} &= -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* f_\pi^4 g_{\underline{27}} (\langle \lambda_{12} L_\mu \rangle \langle \lambda_{31} L^\mu \rangle \\ &\quad - \langle \lambda_{32} L_\mu \rangle \langle \lambda_{11} L^\mu \rangle - 5 \langle \lambda_{32} L_\mu \rangle \langle \lambda_{33} L^\mu \rangle) + \text{h.c.}, \\ \mathcal{L}_{\underline{27}}^{(3/2)} &= -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* f_\pi^4 g_{\underline{27}} (\langle \lambda_{12} L_\mu \rangle \langle \lambda_{31} L^\mu \rangle \\ &\quad + 2 \langle \lambda_{32} L_\mu \rangle \langle \lambda_{11} L^\mu \rangle + \langle \lambda_{32} L_\mu \rangle \langle \lambda_{33} L^\mu \rangle) + \text{h.c.} \end{aligned} \quad (59)$$

The coupling constants $g_{\underline{8}}$ and $g_{\underline{27}}$ can be extracted from the $K \rightarrow \pi\pi$ decay rate and the $\Delta T = 1/2$ enhancement is reflected in these constants.

Now, we are in a position to evaluate the constants $g_{\underline{8}}$ and $g_{\underline{27}}$ from the results of $\langle \mathcal{Q}_i \rangle$. Comparison of Eq.(33) and Eqs.(42-52) with Eqs.(58,59) yields the following results:

$$\begin{aligned} g_{\underline{8}}^{(1/2)} &= -\frac{2}{5} c_1 + \frac{3}{5} c_2 + c_4 + \left(\frac{\langle \bar{q}q \rangle}{M f_\pi^2} - \frac{\langle \bar{q}q \rangle N_c M}{8\pi^2 f_\pi^4} \right) c_6 - \frac{3}{5} c_9 + \frac{2}{5} c_{10}, \\ g_{\underline{27}}^{(1/2)} &= \frac{1}{15} c_1 + \frac{1}{15} c_2 + \frac{1}{10} c_9 + \frac{1}{10} c_{10}, \\ g_{\underline{27}}^{(3/2)} &= 5 g_{\underline{27}}^{(1/2)}. \end{aligned} \quad (60)$$

We employed the Wilson coefficients c_i obtained by Buchalla *et al.* [9] as shown in Table I. There are three different renormalization schemes. The LO denotes the summation of the leading logarithmic terms $\sim \alpha_s \ln(M_W/\mu))^n$, which were mainly done by Vainshtein *et al.* [4,5], Gilman and Wise [6] and Guberina and Peccei [7]. The NDR and HV represent respectively ‘‘Naive dimensional regularization’’ and ‘t Hooft-Veltman scheme [39,40] (see Ref. [9] for details). In the leading order contribution in the large N_c expansion three parameters are involved: the pion decay constant, the quark condensate, and the constituent quark mass. The values of the quark condensate and constituent quark mass are the parameters we can play with. However, these two parameters are to some extent theoretically restricted. The value of the quark condensate lies between $-(300 \text{ MeV})^3 \leq \langle \bar{q}q \rangle / 2 \leq -(200 \text{ MeV})^3$. Larger values give slightly better ratio of the constants $g_{\underline{8}}$ and $g_{\underline{27}}$. The constituent quark

mass is in fact the free parameter of the χ QM. It is known that the value $M \simeq 400$ MeV describes consistently very well the static properties of the baryon [41]. However, in the mesonic sector lower values are often voted [42]. We also find that lower values of the constituent quark mass provide better ratios of the constants. Figure 1 shows the dependence of the g_8/g_{27} on the M . In the NDR scheme M -dependence is stronger than in the other two schemes. One can easily understand this dependence. The parameter M appears in front of the coefficient c_6 in Eq.(60). From Table I we find that the Wilson coefficient c_6 based on the NDR scheme (-0.0022) is larger than in the other two schemes (-0.009) which causes the strong dependence of the ratio g_8/g_{27} on the M in the case of the NDR scheme. Because of the same reason, its dependence on the quark condensate looks very similar, see Fig. 2. In Table II we find that the ratio g_8/g_{27} is almost seven times underestimated, compared to the empirical data ($g_8/g_{27} \simeq 22$ with the counterterms).

From the calculation of the $\langle \mathcal{Q}_{\Delta S=2} \rangle$ in Eq.(52), we easily write the effective $\Delta S = 2$ weak chiral Lagrangian to order $\mathcal{O}(p^2)$:

$$\mathcal{L}_{\text{eff}}^{\Delta S=2, \mathcal{O}(p^4)} = -\frac{G_F^2 M_W^2}{4\pi^2} \mathcal{F}(\lambda_c, \lambda_t, m_c^2, m_t^2, M_W^2) b(\mu) f_\pi^4 \langle \lambda_6 L_\mu \rangle \langle \lambda_6 L^\mu \rangle. \quad (61)$$

B. $\mathcal{O}(N_c)$ and axial-vector coupling corrections

So far we concentrate on the leading order in the large N_c expansion. We now want to introduce the next-to-leading order corrections in the large N_c expansion. The \mathcal{Q}_3 and \mathcal{Q}_5 survive and the additional terms come into existence in the other quark operators. The VEV of the quark operators are obtained in the $\mathcal{O}(N_c)$ order as follows:

$$\langle Q_1 + \mathcal{Q}_1^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \frac{f_\pi^4}{N_c} \left(\frac{3}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{3} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right) \quad (62)$$

$$\langle Q_2 + \mathcal{Q}_2^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \frac{f_\pi^4}{N_c} \left(-\frac{2}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{3} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right) \quad (63)$$

$$\langle Q_3 + \mathcal{Q}_3^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \frac{f_\pi^4}{N_c} \langle \lambda_6 L_\mu L_\mu \rangle \quad (64)$$

$$\langle Q_4 + \mathcal{Q}_4^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = 0 \quad (65)$$

$$\langle Q_5 + \mathcal{Q}_5^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \left(\frac{f_\pi^2 \langle \bar{q}q \rangle}{N_c M} - \frac{\langle \bar{q}q \rangle M}{8\pi^2} \right) \langle \lambda_6 L_\mu L_\mu \rangle \quad (66)$$

$$\langle Q_6 + \mathcal{Q}_6^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = 0 \quad (67)$$

$$\langle \mathcal{Q}_7 + \mathcal{Q}_7^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \left(\frac{3f_\pi^2 \langle \bar{q}q \rangle}{4N_c M} - \frac{3\langle \bar{q}q \rangle M}{32\pi^2} \right) \left(\langle \lambda_6 \partial_\mu U^\dagger \partial_\mu U U^\dagger \hat{Q} U \rangle + \langle \lambda_6 U^\dagger \hat{Q} U \partial_\mu U^\dagger \partial_\mu U \rangle \right) \quad (68)$$

$$\langle \mathcal{Q}_8 + \mathcal{Q}_8^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \frac{3f_\pi^4}{2N_c} \langle \lambda_6 U^\dagger \partial_\mu U \rangle \langle \hat{Q} \partial_\mu U U^\dagger \rangle \quad (69)$$

$$\langle Q_9 + \mathcal{Q}_9^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \frac{f_\pi^4}{N_c} \left(\frac{2}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{2} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right) \quad (70)$$

$$\langle Q_{10} + \mathcal{Q}_{10}^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(N_c)} = \frac{f_\pi^4}{N_c} \left(-\frac{3}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{2} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right). \quad (71)$$

Taking into account the $\mathcal{O}(N_c)$ corrections given above, we get the $g_{\underline{8}}$ and $g_{\underline{27}}$:

$$g_{\underline{8}}^{(\mathcal{O}(N_c^2)+\mathcal{O}(N_c))} = \left(-\frac{2}{5} + \frac{1}{N_c}\frac{3}{5}\right) c_1 + \left(\frac{3}{5} - \frac{1}{N_c}\frac{2}{5}\right) c_2 + \frac{1}{N_c} c_3 + c_4 \\ + \left(\frac{\langle\bar{q}q\rangle}{N_c f_\pi^2 M} - \frac{\langle\bar{q}q\rangle M}{8 f_\pi^4 \pi^2}\right) c_5 + \left(\frac{\langle\bar{q}q\rangle}{f_\pi^2 M} - \frac{N_c \langle\bar{q}q\rangle M}{8 f_\pi^4 \pi^2}\right) c_6 \\ + \left(-\frac{3}{5} + \frac{1}{N_c}\frac{2}{5}\right) c_9 + \left(\frac{2}{5} - \frac{1}{N_c}\frac{3}{5}\right) c_{10} \quad (72)$$

$$g_{\underline{27}}^{(\mathcal{O}(N_c^2)+\mathcal{O}(N_c))} = \left(1 + \frac{1}{N_c}\right) \left(\frac{3}{5} c_1 + \frac{3}{5} c_2 + \frac{9}{10} c_9 + \frac{9}{10} c_{10}\right). \quad (73)$$

Because of the sign in the $1/N_c$ corrections in Eqs.(72,73), we can easily see that the $\mathcal{O}(N_c)$ correction suppresses the octet coupling $g_{\underline{8}}$ while enhancing the eikosiheptaplet coupling $g_{\underline{27}}$. It indicates that the $\mathcal{O}(N_c)$ corrections make the ratio of these two couplings even worse than that with only the leading contribution. As shown in Table III the ratio $g_{\underline{8}}/g_{\underline{27}}$ is completely underestimated.

It is also interesting to consider the effect of the quark axial-vector coupling constant. To be more consistent in the large N_c expansion, we can take into account subleading order couplings in the large N_c in addition to the leading order coupling given by $\bar{\psi}U^{\gamma_5}\psi$. The simplest way of generalizing the χ QM is to introduce the quark axial-vector coupling g_A different from unity [43,44]. In such a case the g_A is known to be smaller than 1. The g_A enters in the effective action given in Eq.(6):

$$S_{\text{eff}}[\pi] = -N_c \text{Tr} \ln (i\bar{\psi} + iMU^{\gamma_5} + i\epsilon_A U^{\gamma_5} \bar{\psi}U^{\gamma_5}), \quad (74)$$

where $\epsilon_A = (1 - g_A)/2$. The understanding of this coupling depends on the specific dynamical assumptions. There are two different arguments about the large N_c behavior of the $1 - g_A^2$. For example, Weinberg argued that $1 - g_A^2$ is of order $\mathcal{O}(1/N_c)$ using the Adler-Weisberger sum rule [45]. Also Dicus *et al.* [46] considered it as $1/N_c$ corrections. On the other hand, Broniowski *et al.* [47] demonstrated that from the Adler-Weisberger sum rule with the reggeized ρ meson exchange $1 - g_A^2$ is of order $\mathcal{O}(N_c^0)$. The new term $i\epsilon_A U^{\gamma_5} \bar{\psi}U^{\gamma_5}$ being considered, the operators $(A_{1,2})_\mu$ can be rewritten as

$$(A_{1,2})_\mu = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + M^2} \sum_{n=0}^{\infty} \left[(\partial^2 - 2ik\partial + M(\bar{\psi}U^{\gamma_5}) + \epsilon_A((\bar{\psi}U^{\gamma_5})(\bar{\psi}U^{\gamma_5}) \right. \\ + U^{-\gamma_5}(\partial^2 U^{\gamma_5}) + 2U^{-\gamma_5}(\partial_\nu U^{\gamma_5})\partial_\nu + 2(\bar{\psi}U^{\gamma_5})U^{-\gamma_5}\bar{\psi} \\ - 2iU^{-\gamma_5}(\partial_\nu U^{\gamma_5})k_\nu - 2i(\bar{\psi}U^{\gamma_5})U^{-\gamma_5}\not{k}) \\ \left. + \epsilon_A^2(\bar{\psi}U^{\gamma_5})(\bar{\psi}U^{\gamma_5})\right) \frac{1}{k^2 + M^2} \Big]^n \\ \times (i\bar{\psi} - iMU^{-\gamma_5} + i\epsilon_A(\bar{\psi}U^{\gamma_5})U^{-\gamma_5} + \not{k}) \gamma_\mu P_{R,L} \Lambda_{1,2}. \quad (75)$$

Since the parameter ϵ_A is tiny, we can safely neglect the ϵ_A^2 terms. Thus, we obtain the following results:

$$\langle \mathcal{Q}_1 + \mathcal{Q}_1^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} = -f_\pi^4 \epsilon_A \left(\frac{\langle\bar{q}q\rangle}{2Mf_\pi^2} + 3 \right)$$

$$\times \left(-\frac{2}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{3} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right), \quad (76)$$

$$\begin{aligned} \langle \mathcal{Q}_2 + \mathcal{Q}_2^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} &= -f_\pi^4 \epsilon_A \left(\frac{\langle \bar{q}q \rangle}{2M f_\pi^2} + 3 \right) \\ &\times \left(\frac{3}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{3} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right), \end{aligned} \quad (77)$$

$$\langle \mathcal{Q}_3 + \mathcal{Q}_3^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} = 0 \quad (78)$$

$$\langle \mathcal{Q}_4 + \mathcal{Q}_4^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} = -f_\pi^4 \epsilon_A \left(\frac{\langle \bar{q}q \rangle}{2M f_\pi^2} + 3 \right) \langle \lambda_6 L_\mu L_\mu \rangle, \quad (79)$$

$$\langle \mathcal{Q}_5 + \mathcal{Q}_5^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} = 0 \quad (80)$$

$$\langle \mathcal{Q}_6 + \mathcal{Q}_6^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} = -f_\pi^4 \epsilon_A \left(\frac{4 \langle \bar{q}q \rangle}{M f_\pi^2} - \frac{3 N_c \langle \bar{q}q \rangle M}{4 \pi^2 f_\pi^2} \right) \langle L_\mu L_\mu \lambda_6 \rangle, \quad (81)$$

$$\langle \mathcal{Q}_7 + \mathcal{Q}_7^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} = -\epsilon_A \left(\frac{3 f_\pi^2 \langle \bar{q}q \rangle}{4M} + \frac{9}{2} f_\pi^4 \right) \langle \lambda_6 U^\dagger \partial_\mu U \rangle \langle \hat{Q} \partial_\mu U U^\dagger \rangle \quad (82)$$

$$\langle \mathcal{Q}_8 + \mathcal{Q}_8^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} = -\epsilon_A \left(\frac{3 f_\pi^2 \langle \bar{q}q \rangle}{M} - \frac{9 N_c \langle \bar{q}q \rangle M}{16 \pi^2} \right), \quad (83)$$

$$\times \left(\langle \lambda_6 \partial_\mu U^\dagger \partial_\mu U U^\dagger \hat{Q} U \rangle + \langle \lambda_6 U^\dagger \hat{Q} U \partial_\mu U^\dagger \partial_\mu U \rangle \right), \quad (84)$$

$$\begin{aligned} \langle \mathcal{Q}_9 + \mathcal{Q}_9^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} &= -f_\pi^4 \epsilon_A \left(\frac{\langle \bar{q}q \rangle}{2M f_\pi^2} + 3 \right) \\ &\times \left(-\frac{3}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{2} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right), \end{aligned} \quad (85)$$

$$\begin{aligned} \langle \mathcal{Q}_{10} + \mathcal{Q}_{10}^\dagger \rangle_{\mathcal{O}(p^2)}^{\mathcal{O}(\epsilon_A)} &= -f_\pi^4 \epsilon_A \left(\frac{\langle \bar{q}q \rangle}{2M f_\pi^2} + 3 \right) \\ &\times \left(\frac{2}{5} \langle \lambda_6 L_\mu L_\mu \rangle + \frac{1}{2} t_{ij;kl} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\mu \rangle \right). \end{aligned} \quad (86)$$

The LECs can be then obtained as follows:

$$\begin{aligned} g_{\underline{8}}^{g_A} &= \left(1 - \epsilon_A \left(\frac{\langle \bar{q}q \rangle}{2M f_\pi^2} + 3 \right) \right) \left(-\frac{2}{5} c_1 + \frac{3}{5} c_2 + c_4 - \frac{3}{5} c_9 + \frac{2}{5} c_{10} \right) \\ &+ \left(\frac{\langle \bar{q}q \rangle}{f_\pi^2 M} - \frac{N_c \langle \bar{q}q \rangle M}{8 f_\pi^4 \pi^2} - \epsilon_A \left(\frac{4 \langle \bar{q}q \rangle}{M f_\pi^2} - \frac{3 N_c \langle \bar{q}q \rangle M}{4 \pi^2 f_\pi^2} \right) \right) c_6 \\ g_{\underline{27}}^{g_A} &= \left(1 - \epsilon_A \left(\frac{\langle \bar{q}q \rangle}{2M f_\pi^2} + 3 \right) \right) \left(\frac{3}{5} c_1 + \frac{3}{5} c_2 + \frac{9}{10} c_9 + \frac{9}{10} c_{10} \right). \end{aligned} \quad (87)$$

Figure 3 shows the dependence of the $g_{\underline{8}}/g_{\underline{27}}$ on the quark axial-vector constant g_A ranging from 0.75 to 1.25. The dependence on the g_A is stronger again in the case of the NDR scheme. To get a reasonable value for the ratio one should choose a large value of g_A which, however, deviates from the physical value $g_A \simeq 0.75$, as easily found from Eq.(87). Thus, corrections from the $\mathcal{O}(N_c)$ and axial-vector coupling constants turn out to be quite useless if one wants to reproduce the empirical data.

The effective $\Delta S = 2$ weak chiral Lagrangian with the $1/N_c$ and g_A corrections is given as follows:

$$\mathcal{L}_{\text{eff}}^{\Delta S=2, \mathcal{O}(p^2)} = -\frac{G_F^2 M_W^2}{4\pi^2} \mathcal{F}(\lambda_c, \lambda_t, m_c^2, m_t^2, M_W^2) b(\mu) \left[f_\pi^4 + \frac{f_\pi^4}{N_c} - \epsilon_A \left(\frac{f_\pi^2 \langle \bar{q}q \rangle}{2M} + 3f_\pi^4 \right) \right] \langle \lambda_6 L_\mu \rangle \langle \lambda_6 L^\mu \rangle. \quad (88)$$

V. NEXT-TO-LEADING ORDER $\mathcal{O}(P^4)$

Although the derivative expansion to order $\mathcal{O}(p^4)$ is straightforward, reducing the number of terms is quite involved. We first can reduce the terms containing higher-order derivatives by using the following identities:

$$U^+ (\partial_\mu \partial_\nu U) = -\frac{1}{2} \{L_\mu, L_\nu\} - \frac{1}{4} i W_{\mu\nu}, \quad (89)$$

$$(\partial_\mu \partial_\nu U^+) U = -\frac{1}{2} \{L_\mu, L_\nu\} + \frac{1}{4} i W_{\mu\nu}, \quad (90)$$

where

$$W_{\mu\nu} = 2(\partial_\mu L_\nu - \partial_\nu L_\mu). \quad (91)$$

We can then compare the reduced set of terms in the $\mathcal{O}(p^4)$ order Lagrangian with that in Ref. [17]. To this end the number of terms can be reduced further by employing the equation of motion for the meson fields in the chiral limit $\partial_\mu L_\mu = 0$ and the identities

$$\frac{1}{8} \langle W_{\mu\nu}^2 \Lambda \rangle = \langle L_\mu L_\nu L_\nu L_\mu \Lambda \rangle - \langle L_\mu L_\nu L_\mu L_\nu \Lambda \rangle, \quad (92)$$

$$\begin{aligned} \frac{1}{4} i \langle L_\mu \Lambda \rangle \langle [W_{\mu\nu}, L_\nu] \Lambda \rangle &= \langle L_\mu L_\nu \Lambda \rangle \langle L_\nu L_\mu \Lambda \rangle - \langle L_\mu L_\nu \Lambda \rangle \langle L_\mu L_\nu \Lambda \rangle \\ &\quad + \langle L_\mu \Lambda \rangle \langle L_\nu L_\mu L_\nu \Lambda \rangle - \frac{1}{2} \langle L_\mu \Lambda \rangle \langle \{L_\mu, L_\nu L_\nu\} \Lambda \rangle, \end{aligned} \quad (93)$$

$$\begin{aligned} \frac{1}{2} i \langle L_\mu \Lambda \rangle \langle W_{\mu\nu} L_\nu \rangle &= \langle L_\mu L_\nu \Lambda \rangle \langle L_\nu L_\mu \Lambda \rangle - \langle L_\mu L_\nu \Lambda \rangle \langle L_\mu L_\nu \Lambda \rangle \\ &\quad + \langle L_\mu \Lambda \rangle \langle L_\nu L_\mu L_\nu \Lambda \rangle - \langle L_\mu \Lambda \rangle \langle L_\nu L_\nu L_\mu \Lambda \rangle, \end{aligned} \quad (94)$$

where Λ denote arbitrary flavor matrices. The identities (92)-(94) can be easily obtained by integration by parts and some trace identities from the Cayley-Hamilton theorem. Decomposing the octet and eikosiheptaplet contributions, we end up with the following results for the vacuum expectation values at $\mathcal{O}(p^4)$ order and leading order in the large N_c expansion:

$$\begin{aligned} \langle \mathcal{Q}_1 + \mathcal{Q}_1^\dagger \rangle_{\mathcal{O}(p^4)} &= \frac{N_c f_\pi^2}{24\pi^2} \left[\frac{4}{5} \langle L_\mu L_\nu L_\nu L_\mu \lambda_6 \rangle - \frac{3}{5} \langle L_\mu \lambda_6 \rangle \langle L_\mu L_\nu L_\nu \rangle \right. \\ &\quad - \frac{3}{5} \epsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_6 \rangle \langle L_\beta L_\gamma L_\delta \rangle \\ &\quad - \frac{2}{3} t_{ijkl} \left(\langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\nu L_\mu \lambda_{kl} \rangle - \langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\mu L_\nu \lambda_{kl} \rangle \right. \\ &\quad \left. \left. + \langle L_\mu \lambda_{ij} \rangle \langle L_\nu L_\mu L_\nu \lambda_{kl} \rangle + \epsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_{ij} \rangle \langle L_\beta L_\gamma L_\delta \lambda_{kl} \rangle \right) \right] \quad (95) \end{aligned}$$

$$\begin{aligned} \langle \mathcal{Q}_2 + \mathcal{Q}_2^\dagger \rangle_{\mathcal{O}(p^4)} = & \frac{N_c f_\pi^2}{24\pi^2} \left[-\frac{6}{5} \langle L_\mu L_\nu L_\nu L_\mu \lambda_6 \rangle + \frac{2}{5} \langle L_\mu \lambda_6 \rangle \langle L_\mu L_\nu L_\nu \rangle \right. \\ & + \frac{2}{5} \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_6 \rangle \langle L_\beta L_\gamma L_\delta \rangle \\ & - \frac{2}{3} t_{ijkl} \left(\langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\nu L_\mu \lambda_{kl} \rangle - \langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\mu L_\nu \lambda_{kl} \rangle \right. \\ & \left. \left. + \langle L_\mu \lambda_{ij} \rangle \langle L_\nu L_\mu L_\nu \lambda_{kl} \rangle + \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_{ij} \rangle \langle L_\beta L_\gamma L_\delta \lambda_{kl} \rangle \right) \right] \quad (96) \end{aligned}$$

$$\langle \mathcal{Q}_3 + \mathcal{Q}_3^\dagger \rangle_{\mathcal{O}(p^4)} = -\frac{N_c f_\pi^2}{24\pi^2} \left[\langle L_\mu \lambda_6 \rangle \langle L_\mu L_\nu L_\nu \rangle + \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_6 \rangle \langle L_\beta L_\gamma L_\delta \rangle \right] \quad (97)$$

$$\langle \mathcal{Q}_4 + \mathcal{Q}_4^\dagger \rangle_{\mathcal{O}(p^4)} = -\frac{N_c f_\pi^2}{12\pi^2} \langle L_\mu L_\nu L_\nu L_\mu \lambda_6 \rangle \quad (98)$$

$$\langle \mathcal{Q}_5 + \mathcal{Q}_5^\dagger \rangle_{\mathcal{O}(p^4)} = \frac{N_c f_\pi^2}{24\pi^2} \left[\langle L_\mu \lambda_6 \rangle \langle L_\mu L_\nu L_\nu \rangle - \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_6 \rangle \langle L_\beta L_\gamma L_\delta \rangle \right] \quad (99)$$

$$\langle \mathcal{Q}_6 + \mathcal{Q}_6^\dagger \rangle_{\mathcal{O}(p^4)} = \left(\frac{N_c^2 M^2}{128\pi^4} - \frac{N_c f_\pi^2}{8\pi^2} + \frac{f_\pi^4}{2M^2} \right) \langle L_\mu L_\mu L_\nu L_\nu \lambda_6 \rangle \quad (100)$$

$$\begin{aligned} \langle \mathcal{Q}_7 + \mathcal{Q}_7^\dagger \rangle_{\mathcal{O}(p^4)} = & -\frac{3N_c f_\pi^2}{48\pi^2} \left[2\langle \lambda_6 \partial_\mu U^\dagger \partial_\nu U \rangle \langle \hat{Q} \partial_\nu U \partial_\mu U^\dagger \rangle - 2\langle \lambda_6 \partial_\mu U^\dagger \partial_\nu U \rangle \langle \hat{Q} \partial_\mu U \partial_\nu U^\dagger \rangle \right. \\ & + \langle \lambda_6 U^\dagger \partial_\mu U \rangle \langle \hat{Q} \partial_\nu U \partial_\mu U^\dagger \partial_\nu U U^\dagger \rangle + \langle \lambda_6 U^\dagger \partial_\nu U \partial_\mu U^\dagger \partial_\nu U \rangle \langle \hat{Q} \partial_\mu U U^\dagger \rangle \\ & \left. - \varepsilon_{\alpha\beta\gamma\delta} \left(\langle \lambda_6 U^\dagger \partial_\alpha U \rangle \langle \hat{Q} \partial_\beta U \partial_\gamma U^\dagger \partial_\delta U U^\dagger \rangle + \langle \lambda_6 U^\dagger \partial_\alpha U \partial_\beta U^\dagger \partial_\gamma U \rangle \langle \hat{Q} \partial_\delta U U^\dagger \rangle \right) \right] \quad (101) \end{aligned}$$

$$\begin{aligned} \langle \mathcal{Q}_8 + \mathcal{Q}_8^\dagger \rangle_{\mathcal{O}(p^4)}^{(\mathcal{N}_c^2)} = & \left(\frac{3N_c^2 M^2}{256\pi^4} - \frac{3N_c f_\pi^2}{16\pi^2} + \frac{3f_\pi^4}{4M^2} \right) \langle \lambda_6 \partial_\mu U^\dagger \partial_\mu U U^\dagger \hat{Q} U \partial_\nu U^\dagger \partial_\nu U \rangle \\ & + \frac{3N_c \langle \bar{q} q \rangle}{192M\pi^2} \left[\langle \lambda_6 \partial_\mu U^\dagger \partial_\nu U \partial_\mu U^\dagger \partial_\nu U U^\dagger \hat{Q} U \rangle + \langle \lambda_6 U^\dagger \hat{Q} U \partial_\mu U^\dagger \partial_\nu U \partial_\mu U^\dagger \partial_\nu U \rangle \right. \\ & - \langle \lambda_6 \partial_\mu U^\dagger \partial_\nu U \partial_\nu U^\dagger \partial_\mu U U^\dagger \hat{Q} U \rangle - \langle \lambda_6 U^\dagger \hat{Q} U \partial_\mu U^\dagger \partial_\nu U \partial_\nu U^\dagger \partial_\mu U \rangle \\ & + \frac{1}{2} i \langle \lambda_6 \{ \partial_\mu U^\dagger \partial_\nu U, W_{\mu\nu} \} U^\dagger \hat{Q} U \rangle - \frac{1}{2} i \langle \lambda_6 U^\dagger \hat{Q} U \{ \partial_\mu U^\dagger \partial_\nu U, W_{\mu\nu} \} \rangle \\ & + \frac{1}{8} \langle \lambda_6 W_{\mu\nu}^2 U^\dagger \hat{Q} U \rangle + \frac{1}{8} \langle \lambda_6 U^\dagger \hat{Q} U W_{\mu\nu}^2 \rangle \\ & \left. - 2\varepsilon_{\alpha\beta\gamma\delta} \left(\langle \lambda_6 \partial_\alpha U^\dagger \partial_\beta U \partial_\gamma U^\dagger \partial_\delta U U^\dagger \hat{Q} U \rangle - \langle \lambda_6 U^\dagger \hat{Q} U \partial_\alpha U^\dagger \partial_\beta U \partial_\gamma U^\dagger \partial_\delta U \rangle \right) \right] \quad (102) \end{aligned}$$

$$\begin{aligned} \langle \mathcal{Q}_9 + \mathcal{Q}_9^\dagger \rangle_{\mathcal{O}(p^4)} = & \frac{N_c f_\pi^2}{24\pi^2} \left[\frac{6}{5} \langle L_\mu L_\nu L_\nu L_\mu \lambda_6 \rangle - \frac{2}{5} \langle L_\mu \lambda_6 \rangle \langle L_\mu L_\nu L_\nu \rangle \right. \\ & - \frac{2}{5} \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_6 \rangle \langle L_\beta L_\gamma L_\delta \rangle \\ & - t_{ijkl} \left(\langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\nu L_\mu \lambda_{kl} \rangle - \langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\mu L_\nu \lambda_{kl} \rangle \right. \\ & \left. \left. + \langle L_\mu \lambda_{ij} \rangle \langle L_\nu L_\mu L_\nu \lambda_{kl} \rangle + \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_{ij} \rangle \langle L_\beta L_\gamma L_\delta \lambda_{kl} \rangle \right) \right] \quad (103) \end{aligned}$$

$$\begin{aligned} \langle \mathcal{Q}_{10} + \mathcal{Q}_{10}^\dagger \rangle_{\mathcal{O}(p^4)} = & \frac{N_c f_\pi^2}{24\pi^2} \left[-\frac{4}{5} \langle L_\mu L_\nu L_\nu L_\mu \lambda_6 \rangle + \frac{3}{5} \langle L_\mu \lambda_6 \rangle \langle L_\mu L_\nu L_\nu \rangle \right. \\ & + \frac{3}{5} \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_6 \rangle \langle L_\beta L_\gamma L_\delta \rangle \\ & \left. - t_{ijkl} \left(\langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\nu L_\mu \lambda_{kl} \rangle - \langle L_\mu L_\nu \lambda_{ij} \rangle \langle L_\mu L_\nu \lambda_{kl} \rangle \right) \right] \end{aligned}$$

$$+ \langle L_\mu \lambda_{ij} \rangle \langle L_\nu L_\mu L_\nu \lambda_{kl} \rangle + \varepsilon_{\alpha\beta\gamma\delta} \langle L_\alpha \lambda_{ij} \rangle \langle L_\beta L_\gamma L_\delta \lambda_{kl} \rangle \Big] \quad (104)$$

Comparing Eqs.(95)-(104) with the effective $\Delta S = 1$ weak chiral Lagrangian to order $\mathcal{O}(p^4)$ in χ PT given by Ecker *et al.* [19] and Esposito-Farèse [18] (presented in Minkowski space):

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{\Delta S=1, \mathcal{O}(p^4)} = & -\frac{G_F}{\sqrt{2}} V_{\text{ud}} V_{\text{us}}^* f_\pi^2 \Big[\left(N_1^{(8)} \langle \lambda_6 L_\mu L^\mu L_\nu L^\nu \rangle + N_2^{(8)} \cdot \langle \lambda_6 L_\mu L^\nu L_\nu L^\mu \rangle \right. \\ & + N_3^{(8)} \langle \lambda_6 L_\mu L_\nu \rangle \langle L^\mu L^\nu \rangle + N_4^{(8)} \langle \lambda_6 L_\mu \rangle \langle L^\mu L_\nu L^\nu \rangle \\ & + N_{28}^{(8)} i \epsilon_{\mu\nu\rho\delta} \langle \lambda_6 L^\mu \rangle \langle L^\nu L^\rho L^\delta \rangle \Big) \\ & + \frac{5}{9} t_{ijkl} \left(N_1^{(27)} \langle \lambda_{ij} L_\mu L^\mu \rangle \langle \lambda_{kl} L_\nu L^\nu \rangle + N_2^{(27)} \langle \lambda_{ij} L_\mu L_\nu \rangle \langle \lambda_{kl} L^\mu L^\nu \rangle \right. \\ & + N_3^{(27)} \langle \lambda_{ij} L_\mu L_\nu \rangle \langle \lambda_{kl} L^\nu L^\mu \rangle + N_4^{(27)} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} L_\nu L^\mu L^\nu \rangle \\ & + N_5^{(27)} \langle \lambda_{ij} L_\mu \rangle \langle \lambda_{kl} \{ L^\mu, L_\nu L^\nu \} \rangle + N_6^{(27)} \langle L_\mu L^\mu \rangle \langle \lambda_{ij} L_\nu \rangle \langle \lambda_{kl} L^\nu \rangle \\ & \left. + N_{20}^{(27)} i \epsilon_{\mu\nu\rho\delta} \langle \lambda_{ij} L^\mu L^\nu \rangle \langle \lambda_{kl} L^\rho L^\delta \rangle + N_{21}^{(27)} i \epsilon_{\mu\nu\rho\delta} \langle \lambda_{ij} L^\mu \rangle \langle \lambda_{kl} L^\nu L^\rho L^\delta \rangle \right) \Big], \quad (105) \end{aligned}$$

we derive the LECs in the case of the leading order in the large N_c expansion:

$$N_1^{(8)} = \left(-\frac{N_c^2 M^2}{128\pi^4 f_\pi^2} + \frac{N_c}{8\pi^2} - \frac{f_\pi^2}{2M^2} \right) c_6, \quad (106)$$

$$N_2^{(8)} = \frac{N_c}{60\pi^2} \left(-2c_1 + 3c_2 + 5c_4 - 3c_9 + 2c_{10} \right), \quad (107)$$

$$N_3^{(8)} = 0, \quad (108)$$

$$N_4^{(8)} = \frac{N_c}{60\pi^2} \left(\frac{3}{2}c_1 - c_2 + \frac{5}{2}c_3 - \frac{5}{2}c_5 + c_9 - \frac{3}{2}c_{10} \right), \quad (109)$$

$$N_{28}^{(8)} = \frac{N_c}{60\pi^2} \left(-\frac{3}{2}c_1 + c_2 - \frac{5}{2}c_3 - \frac{5}{2}c_5 - c_9 + \frac{3}{2}c_{10} \right), \quad (110)$$

$$N_1^{(27)} = N_5^{(27)} = N_6^{(27)} = N_{20}^{(27)} = 0, \quad (111)$$

$$N_2^{(27)} = -N_3^{(27)} = -N_4^{(27)} = N_{21}^{(27)} = \frac{N_c}{60\pi^2} \left(-3c_1 - 3c_2 - \frac{9}{2}c_9 - \frac{9}{2}c_{10} \right). \quad (112)$$

As noted by G. Ecker *et al.* [19], the LECs $N_1^{(8)}$, $N_2^{(8)}$, $N_3^{(8)}$ and $N_4^{(8)}$ contribute to the process $K \rightarrow 3\pi$ while $N_{28}^{(8)}$ does to the radiative K -decays. In particular, the $N_{28}^{(8)}$ is related to the chiral anomaly [53]. The numerical results can be found in Table II. Note that the LECs in the eikosiheptaplet can assume only two values.

Taking into account the $1/N_c$ corrections, the LECs are extracted as follows:

$$\begin{aligned} N_1^{(8)} = & \left(-\frac{N_c^2 M^2}{128\pi^4 f_\pi^2} + \frac{N_c}{8\pi^2} - \frac{f_\pi^2}{2M^2} \right) c_6 \\ & + \left(-\frac{N_c M^2}{128\pi^4 f_\pi^2} + \frac{1}{8\pi^2} - \frac{f_\pi^2}{2N_c M^2} \right) c_5, \quad (113) \end{aligned}$$

$$\begin{aligned} N_2^{(8)} = & \frac{N_c}{60\pi^2} \left(\left(-2 + \frac{1}{N_c} 3 \right) c_1 + \left(3 - \frac{1}{N_c} 2 \right) c_2 + \frac{1}{N_c} 5c_3 + 5c_4 \right. \\ & \left. + \left(-3 + \frac{1}{N_c} 2 \right) c_9 + \left(2 - \frac{1}{N_c} 3 \right) c_{10} \right), \quad (114) \end{aligned}$$

$$N_3^{(8)} = 0, \quad (115)$$

$$N_4^{(8)} = \frac{N_c}{60\pi^2} \left(\left(\frac{3}{2} - \frac{1}{N_c} \right) c_1 + \left(-1 + \frac{1}{N_c} \frac{3}{2} \right) c_2 + \frac{5}{2} c_3 + \frac{1}{N_c} \frac{5}{2} c_4 - \frac{5}{2} c_5 - \frac{1}{N_c} \frac{5}{2} c_6 + \left(1 - \frac{1}{N_c} \frac{3}{2} \right) c_9 + \left(-\frac{3}{2} + \frac{1}{N_c} \right) c_{10} \right), \quad (116)$$

$$N_{28}^{(8)} = \frac{N_c}{60\pi^2} \left(\left(-\frac{3}{2} + \frac{1}{N_c} \right) c_1 + \left(1 - \frac{1}{N_c} \frac{3}{2} \right) c_2 - \frac{5}{2} c_3 - \frac{1}{N_c} \frac{5}{2} c_4 - \frac{5}{2} c_5 - \frac{1}{N_c} \frac{5}{2} c_6 + \left(-1 + \frac{1}{N_c} \frac{3}{2} \right) c_9 + \left(\frac{3}{2} - \frac{1}{N_c} \right) c_{10} \right), \quad (117)$$

$$N_1^{(27)} = N_5^{(27)} = N_6^{(27)} = N_{20}^{(27)} = 0, \quad (118)$$

$$N_2^{(27)} = -N_3^{(27)} = -N_4^{(27)} = N_{21}^{(27)} = \frac{N_c}{60\pi^2} \left(1 + \frac{1}{N_c} \right) \left(-3c_1 - 3c_2 - \frac{9}{2}c_9 - \frac{9}{2}c_{10} \right). \quad (119)$$

Table III shows the effect of the $\mathcal{O}(N_c)$ corrections.

The effective $\Delta S = 2$ weak chiral Lagrangian is obtained as

$$\mathcal{L}_{\text{eff}}^{\Delta S=2, \mathcal{O}(p^4)} = -\frac{G_F^2 M_W^2}{4\pi^2} \mathcal{F}(\lambda_c, \lambda_t, m_c^2, m_t^2, M_W^2) b(\mu) \frac{N_c f_\pi^2}{12\pi^2} (\langle \lambda_6 L_\mu L_\nu \rangle \langle \lambda_6 L^\nu L^\mu \rangle - \langle \lambda_6 L_\mu L_\nu \rangle \langle \lambda_6 L^\mu L_\nu \rangle + \langle \lambda_6 L_\mu \rangle \langle \lambda_6 L_\nu L^\mu L^\nu \rangle - i\epsilon_{\mu\nu\alpha\beta} \langle \lambda_6 L^\mu \rangle \langle \lambda_6 L^\nu L^\alpha L^\beta \rangle). \quad (120)$$

The above formulae present the final result for the effective weak chiral Lagrangian for $\Delta S = 1$ and $\Delta S = 2$. It is constructed in a way to be used in χ PT without further treatment, since all LECs are explicitly given. The result is a strict outcome of the χ QM and the expansion of its Lagrangian in powers of the momentum and in the number of colors. It is assumed, however, that no external fields are present.

VI. SUMMARY AND CONCLUSIONS

The aim of the present work has been to derive the effective $\Delta S = 1, 2$ effective weak chiral Lagrangian with its low energy constants from the chiral quark model using the weak effective action of Buchalla, Buras and Lautenbacher [9]. In leading order in the large N_c expansion the Lagrangian is already known. The final result of our investigations is the effective weak chiral Lagrangian in next-to-leading order in N_c and to fourth order in the momentum.

As is already known, the contribution of leading order in the large N_c expansion to the ratio g_8/g_{27} is heavily underestimated in this model. As we show in the present paper the inclusion of the next-to-leading order in the large N_c expansion does not help to improve this result. Hence on the present level of formalism and without further improvements the chiral quark model does not provide low energy constants which can directly be used in chiral perturbation theory for weak processes. Of course such a conclusion can finally only be drawn if a few actual observables have been calculated in chiral perturbation theory by using the above effective weak chiral Lagrangian. However, since the chiral quark model fails heavily in reproducing the ratio g_8/g_{27} we do not have much hope that this will work.

Actually Antonelli *et al.* [24] have added to the lowest order in N_c certain corrections from the gluon condensate known as of order $\mathcal{O}(\alpha_s N_c)$ [48] in order to change the ratio

g_8/g_{27} . They have shown that the $\mathcal{O}(\alpha_s N_c)$ corrections indeed improve numerically the $\Delta T = 1/2$ enhancement. However, in our view there is an important caveat. In fact, the $\mathcal{O}(\alpha_s N_c)^2$ is of the same order as $\mathcal{O}(\alpha_s N_c)$. The latter corrections were neglected by the authors of Ref. [48] hoping that they might be smaller since they involve condensates of higher dimensions. In view of the large size of those corrections such an argument requires further substantiation, even though the numerical results are improved. In this paper no attempt was done to obtain gluonic corrections to the next-to-leading order in N_c .

At the present level of investigation we see the following ways of investigations, which might improve the low energy constants of the chiral quark model.

First: The chiral quark model has been derived from QCD by Diakonov and Petrov [49–52] by assuming a gluonic vacuum configuration which consists of a dilute gas of interacting instantons and anti-instantons. As a result of this approach the constituent quark mass in the chiral quark model is momentum-dependent and it is only an approximation to replace this by a regularization prescription with a properly chosen cut-off parameter. Thus it is interesting to investigate how far the present results change if such a momentum-dependent constituent mass is used. Such an investigation is even necessary if one wants to exploit fully the chiral quark model.

Second: All the results in the present paper are based on the assumption that the effective weak Hamiltonian of Buchalla, Buras and Lautenbacher can be used in connection with the chiral quark model. This, however, is not that clear. If one considers the derivation of the chiral quark model from QCD by Diakonov and Petrov the renormalization point of the model is around 600 MeV corresponding to the average size of the instantons of 0.3 fm and the average distance of 1 fm. The Wilson coefficients of the effective weak Hamiltonian are evaluated at a scale of 1 GeV and it is not obvious if they can be used without further change at 600 MeV. Suggestions for investigations in this direction have recently been given in [54,25,55].

Actually in our next investigations we will follow the first suggestion and will incorporate the momentum-dependent quark mass in the chiral quark model. Such a procedure links the Lagrangian of the chiral quark model to QCD and, perhaps, the results will be improved.

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Figures

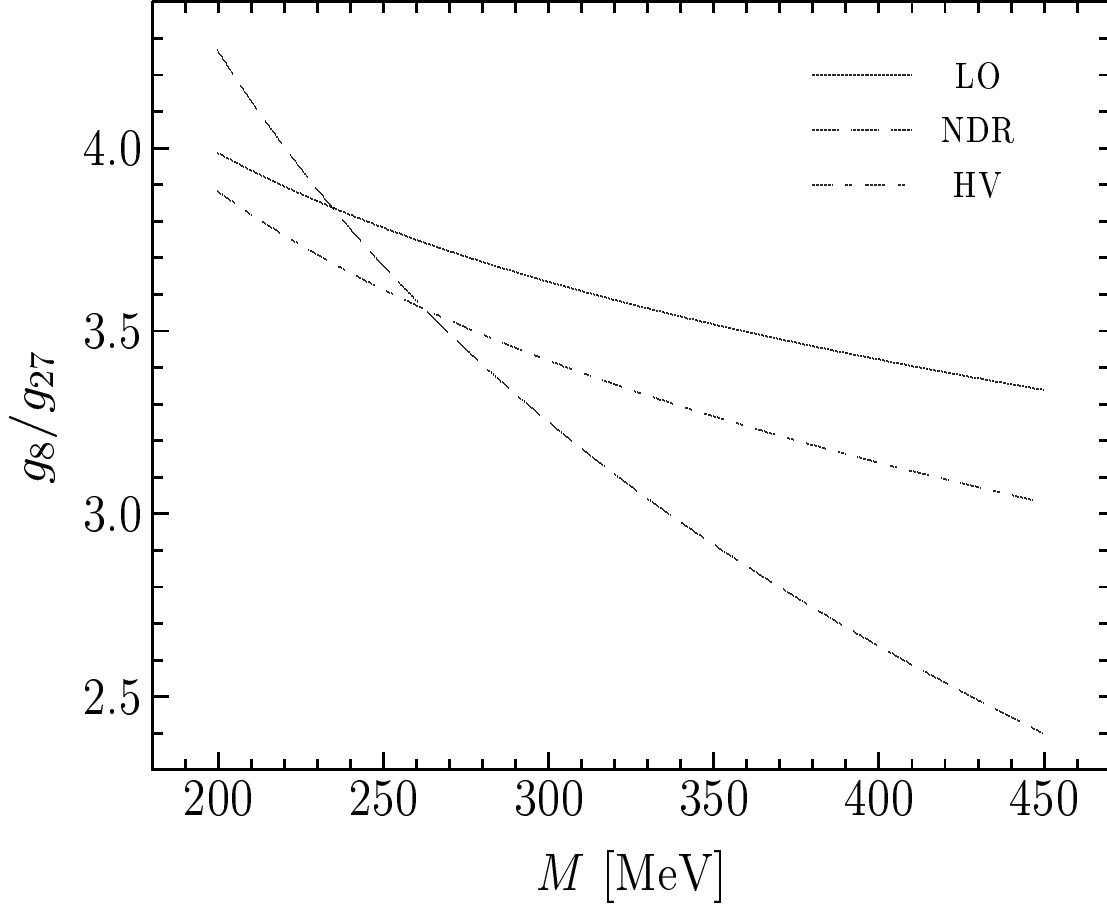


Fig.1: Dependence of the g_8/g_{27} on the M . The solid curve denotes the LO renormalization scheme in Ref. [9], while the dashed curve and dot-dashed one stand for the NDR and the HV schemes, respectively. The value of the quark condensate $\langle \bar{q}q \rangle/2 = -(250 \text{ MeV})^3$ is used.

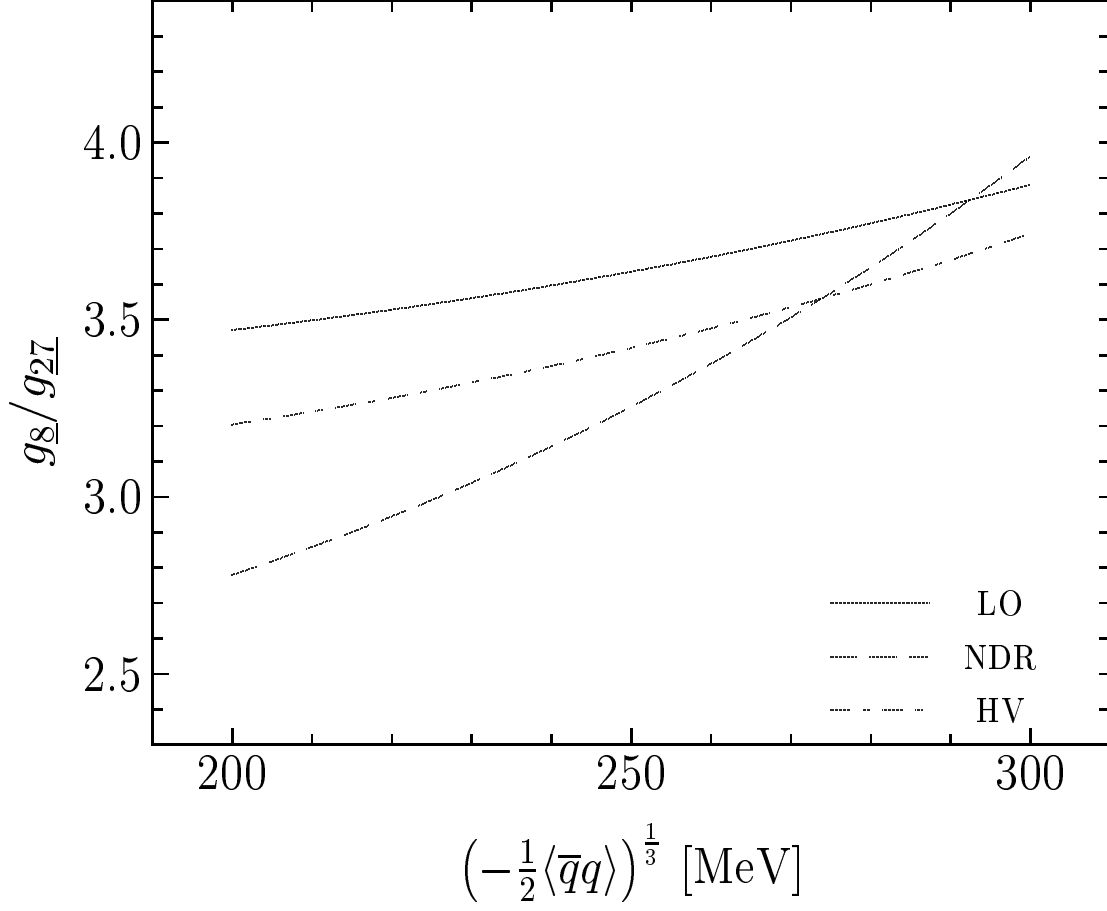


Fig.2: Dependence of the g_8/g_{27} on the $\langle\bar{q}q\rangle$. The solid curve denotes the LO renormalization scheme in Ref. [9], while the dashed curve and dot-dashed one stand for the NDR and the HV schemes, respectively. The value of the constituent quark mass $M = 300$ MeV is used.

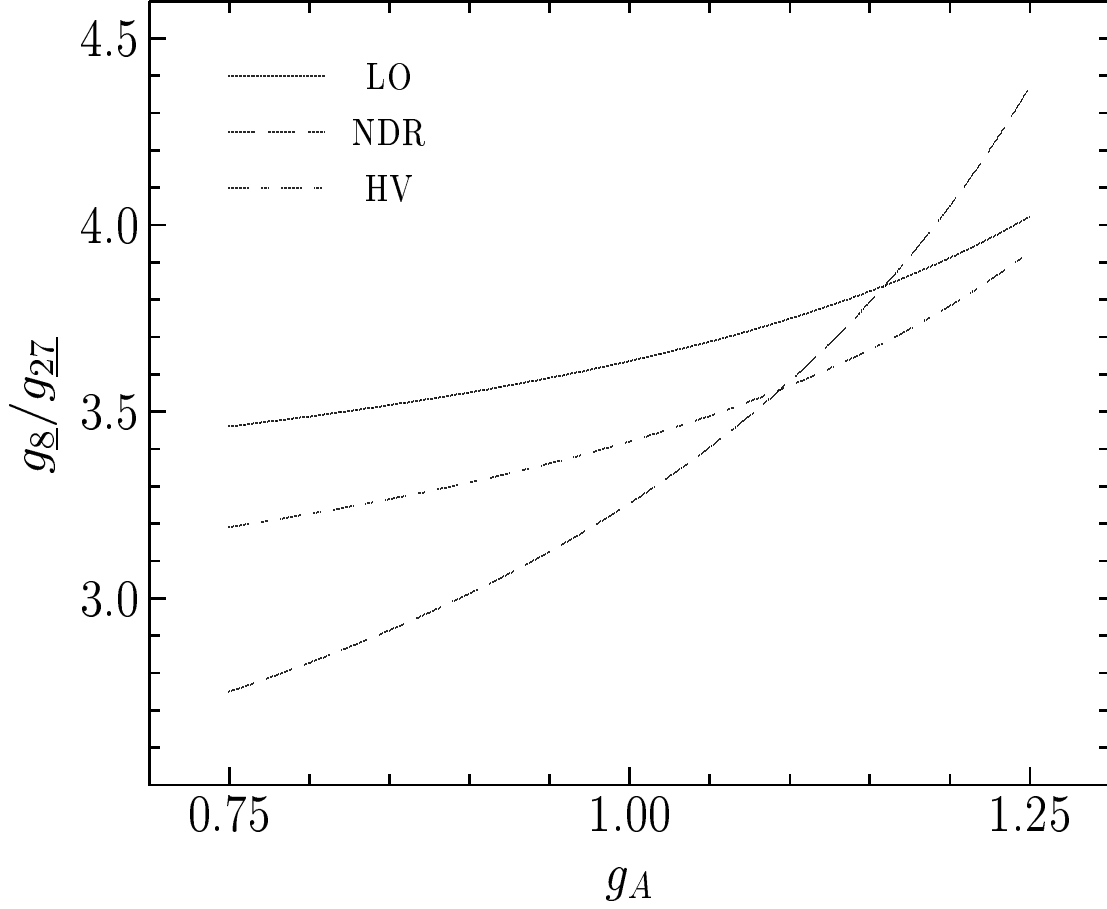


Fig.3: Dependence of the g_8/g_{27} on the quark axial-vector constant g_A . The solid curve denotes the LO renormalization scheme in Ref. [9], while the dashed curve and dot-dashed one stand for the NDR and the HV schemes, respectively. The value of the constituent quark mass $M = 300$ MeV is used and the quark condensate $\langle\bar{q}q\rangle/2 = -(250 \text{ MeV})^3$ is employed.

TABLES

TABLE I. Wilson coefficients at $\mu = 1$ GeV. c_i can be obtained by the relation $c_i(\mu) = z_i(\mu) + \tau y_i(\mu)$ which are provided by Ref. [9].

	$\Lambda_{\overline{\text{MS}}}^{(4)} = 215$ MeV			$\Lambda_{\overline{\text{MS}}}^{(4)} = 325$ MeV			$\Lambda_{\overline{\text{MS}}}^{(4)} = 435$ MeV		
Scheme	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
c_1	-0.607	-0.409	-0.494	-0.748	-0.509	-0.640	-0.907	-0.625	-0.841
c_2	1.333	1.212	1.267	1.433	1.278	1.371	1.552	1.361	1.525
c_3	0.003	0.008	0.004	0.004	0.013	0.007	0.006	0.023	0.015
c_4	-0.008	-0.022	-0.010	-0.012	-0.035	-0.017	-0.017	-0.058	-0.029
c_5	0.003	0.006	0.003	0.004	0.008	0.004	0.005	0.009	0.005
c_6	-0.009	-0.022	-0.009	-0.013	-0.035	-0.014	-0.018	-0.059	-0.025
c_7/α	0.004	0.003	-0.003	0.008	0.011	-0.002	0.011	0.021	-0.001
c_8/α	0	0.008	0.006	0.001	0.014	0.010	0.001	0.027	0.017
c_9/α	0.006	0.008	0.001	0.009	0.019	0.006	0.013	0.035	0.012
c_{10}/α	0	-0.005	-0.006	-0.002	-0.008	-0.010	-0.002	-0.015	-0.018

TABLE II. The low energy constants in $\mathcal{O}(N_c^2)$ order. The Wilson coefficients are from Ref. [9] as shown in Table I. $M = 300$ MeV and $\langle \bar{q}q \rangle = -2 \cdot (250 \text{ MeV})^3$ are used.

	$\Lambda_{\overline{\text{MS}}}^{(4)} = 215$ MeV			$\Lambda_{\overline{\text{MS}}}^{(4)} = 325$ MeV			$\Lambda_{\overline{\text{MS}}}^{(4)} = 435$ MeV		
Scheme	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
g_8	1.100	1.029	1.013	1.241	1.190	1.163	1.407	1.437	1.404
g_{27}	0.436	0.482	0.464	0.411	0.461	0.439	0.387	0.442	0.410
g_8/g_{27}	2.524	2.135	2.184	3.019	2.578	2.652	3.636	3.254	3.421
$N_1^{(8)} \cdot 10^3$	0.16	0.39	0.16	0.23	0.61	0.24	0.31	1.03	0.44
$N_2^{(8)} \cdot 10^3$	26.21	22.01	24.01	29.05	23.69	26.89	32.35	25.54	30.96
$N_3^{(8)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_4^{(8)} \cdot 10^3$	-11.37	-9.22	-10.16	-12.94	-10.28	-11.77	-14.74	-11.47	-13.99
$N_{28}^{(8)} \cdot 10^3$	11.29	9.07	10.08	12.84	10.08	11.67	14.62	11.24	13.86
$N_1^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_2^{(27)} \cdot 10^3$	-11.03	-12.20	-11.75	-10.41	-11.69	-11.11	-9.80	-11.19	-10.39
$N_3^{(27)} \cdot 10^3$	11.03	12.20	11.75	10.41	11.69	11.11	9.80	11.19	10.39
$N_4^{(27)} \cdot 10^3$	11.03	12.20	11.75	10.41	11.69	11.11	9.80	11.19	10.39
$N_5^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_6^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_{20}^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_{21}^{(27)} \cdot 10^3$	-11.03	-12.20	-11.75	-10.41	-11.69	-11.11	-9.80	-11.19	-10.39

TABLE III. Low energy constants with $\mathcal{O}(N_c^2)$ and $\mathcal{O}(N_c)$ contributions. The Wilson coefficients are from Ref. [9] as shown in Table I. $M = 300 \text{ MeV}$ and $\langle \bar{q}q \rangle = -2 \cdot (250 \text{ MeV})^3$ are used.

	$\Lambda_{\overline{\text{MS}}}^{(4)} = 215 \text{ MeV}$			$\Lambda_{\overline{\text{MS}}}^{(4)} = 325 \text{ MeV}$			$\Lambda_{\overline{\text{MS}}}^{(4)} = 435 \text{ MeV}$		
Scheme	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
g_8	0.794	0.773	0.739	0.892	0.902	0.845	1.009	1.117	1.025
g_{27}	0.581	0.642	0.618	0.548	0.615	0.585	0.516	0.589	0.547
g_8/g_{27}	1.368	1.204	1.196	1.628	1.467	1.445	1.955	1.896	1.874
$N_1^{(8)} \cdot 10^3$	0.14	0.35	0.14	0.20	0.57	0.22	0.29	0.98	0.41
$N_2^{(8)} \cdot 10^3$	18.66	15.91	17.26	20.46	16.91	19.08	22.56	17.98	21.68
$N_3^{(8)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_4^{(8)} \cdot 10^3$	-6.96	-5.46	-6.12	-8.05	-6.18	-7.23	-9.28	-6.96	-8.72
$N_{28}^{(8)} \cdot 10^3$	6.96	5.50	6.12	8.05	6.27	7.25	9.30	7.23	8.81
$N_1^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_2^{(27)} \cdot 10^3$	-14.71	-16.27	-15.66	-13.88	-15.59	-14.81	-13.07	-14.92	-13.86
$N_3^{(27)} \cdot 10^3$	14.71	16.27	15.66	13.88	15.59	14.81	13.07	14.92	13.86
$N_4^{(27)} \cdot 10^3$	14.71	16.27	15.66	13.88	15.59	14.81	13.07	14.92	13.86
$N_5^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_6^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_{20}^{(27)} \cdot 10^3$	0	0	0	0	0	0	0	0	0
$N_{21}^{(27)} \cdot 10^3$	-14.71	-16.27	-15.66	-13.88	-15.59	-14.81	-13.07	-14.92	-13.86